



Discrete Mathematics 2025 Spring



魏可佶 kejiwei@tongji.edu.cn



- 6.1 Basic Concepts of Graphs
- 6.2 Connectivity of Graphs
- 6.3 Matrix Representations of Graphs
- 6.4 Several Special Types of Graphs

■ 6.4.1 Bipartite Graphs

Necessary and sufficient conditions for a graph to be bipartite
matching, maximal matching, maximum matching, complete
matching, perfect matching

■ 6.4.2 Eulerian Graphs

Eulerian circuits (paths) and their necessary and sufficient
conditions for existence

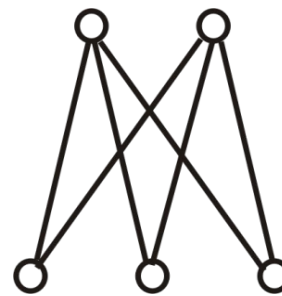
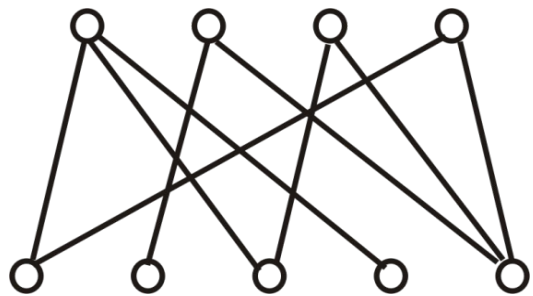
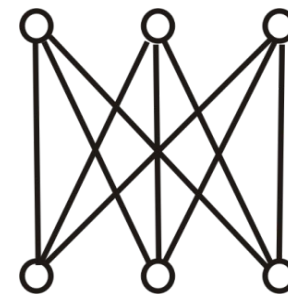
■ 6.4.3 Hamiltonian Graphs

Hamiltonian circuits (paths) and the necessary and sufficient
conditions for their existence

■ 6.4.4 Planar Graphs

↳ Bipartite Graph and Complete Bipartite Graph

- Let $G=\langle V,E\rangle, G=\langle V,E\rangle$ be an undirected graph. If it is possible to partition V into two sets V_1 and V_2 such that: $V_1\cup V_2=V$, $V_1\cap V_2=\emptyset$, Each edge in G has one endpoint in V_1 and the other in V_2 , then G is called a **bipartite graph**, denoted as $\langle V_1, V_2, E \rangle$, and V_1 and V_2 are called **complementary vertex subsets**.
- Furthermore, if G is a simple graph and every vertex in V_1 is adjacent to every vertex in V_2 , then G is called a **complete bipartite graph**, denoted as $K_{r,s}$, where $r=|V_1|$ and $s=|V_2|$.

 $K_{2,3}$  $K_{3,3}$

↳ Cycle Characterization of Bipartite Graphs

■ **Theorem 6.5:** An undirected graph $G = \langle V, E \rangle$ is a bipartite graph if and only if it contains no odd-length cycles.

■ **Proof:**

(1) **Necessity:** Let $G = \langle V_1, V_2, E \rangle$ be a bipartite graph. Each edge can only connect V_1 to V_2 or V_2 to V_1 , so any cycle in G must have even length.

(2) **Sufficiency:** Assume G has at least one edge and is connected.

Take any vertex u , and define:

$V_1 = \{v \mid v \in V \text{ and the distance from } v \text{ to } u \text{ is even}\}$

$V_2 = \{v \mid v \in V \text{ and the distance from } v \text{ to } u \text{ is odd}\}$

Then, $V_1 \cup V_2 = V$, $V_1 \cap V_2 = \emptyset$.

■ Proof:

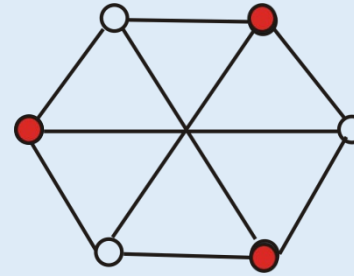
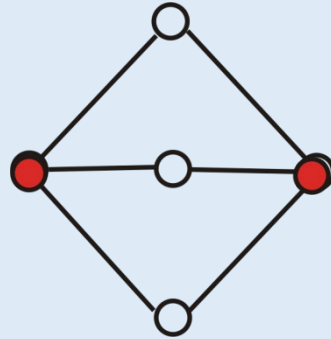
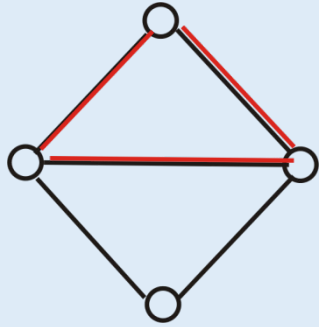
(2) **Sufficiency:** Assume G has at least one edge and is connected. Take any vertex u , and define:

$$V_1 = \{v \mid v \in V \text{ and the distance from } v \text{ to } u \text{ is even}\}$$

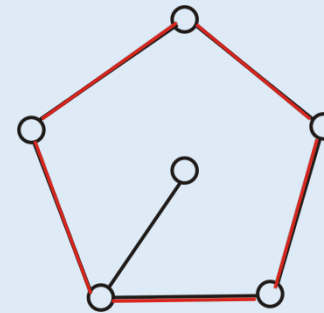
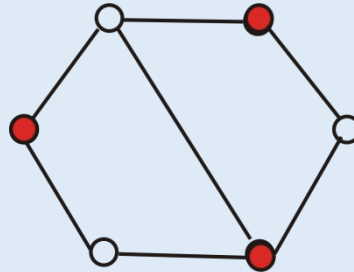
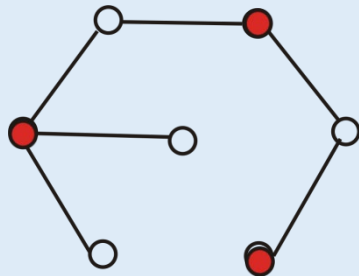
$$V_2 = \{v \mid v \in V \text{ and the distance from } v \text{ to } u \text{ is odd}\}$$

Then, $V_1 \cup V_2 = V$, $V_1 \cap V_2 = \emptyset$.

- First, we prove that no two vertices in V_1 are adjacent. Suppose there exist $s, t \in V_1$ such that $e = (s, t) \in E$. Let Γ_1 and Γ_2 be the shortest paths from u to s and u to t , respectively. Then, $\Gamma_1 \cup e \cup \Gamma_2$ forms a cycle of odd length, which contradicts the assumption.
- Similarly, we can prove that no two vertices in V_2 are adjacent.



Non-bipartite graph



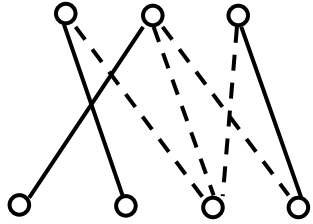
Non-bipartite graph

↳ Matchings and Complete Matchings in Bipartite Graphs

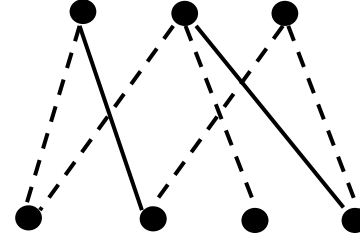
- **Definition 6.11:** Let $G = \langle V_1, V_2, E \rangle$ be a bipartite graph, where $E' \subseteq E$. If the edges in E' are pairwise non-adjacent, then E' is called a **matching** in G . If adding any edge to E' results in a set of edges that is no longer a matching, then E' is called a **maximal matching** in G . The matching in G with the maximum number of edges is called the **maximum matching** of G .
- Moreover, suppose $|V_1| \leq |V_2|$ and E' is a matching in G . If $|E'| = |V_1|$, then E' is called a **complete matching** from V_1 to V_2 .
- When $|V_1| = |V_2|$, a complete matching is called a **perfect matching**.

6.4.1 Bipartite Graphs

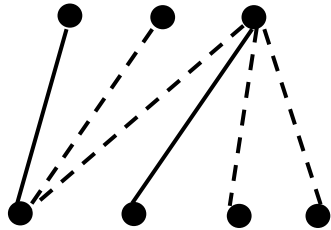
↳ Matchings and Complete Matchings in Bipartite Graphs(e.g.)



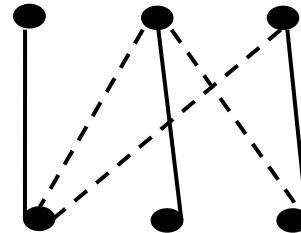
Maximum Matching and
Complete Matching



Maximal Matching



Maximum Matching



Perfect Matching

■ Theorem 6.6 (Hall's Theorem):

Let $G = \langle V_1, V_2, E \rangle$ be a bipartite graph with $|V_1| = |V_2|$.

There **exists a complete matching** from V_1 to V_2 in G if and only if, for any k (where $1 \leq k \leq |V_1|$), the set of k vertices in V_1 is adjacent to at least k vertices in V_2 (the **distinctness condition**).

■ Theorem 6.7:

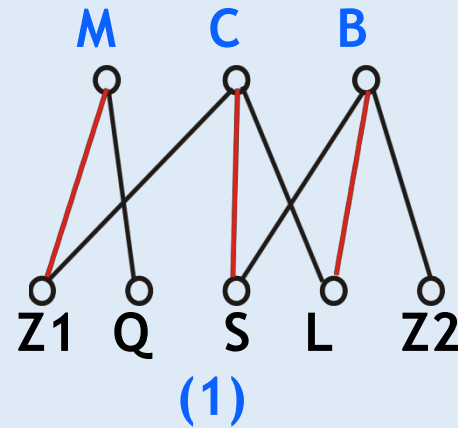
Let $G = \langle V_1, V_2, E \rangle$ be a bipartite graph with $|V_1| \leq |V_2|$. If there exists a positive integer t such that each vertex in V_1 is connected to at least t edges, and each vertex in V_2 is connected to at most t edges (the **t-condition**), then there **exists a complete matching** from V_1 to V_2 in G .

■ Example:

A middle school has three extracurricular activity groups: the Math Group, the Computer Group, and the Biology Group. There are five students: Zhao, Qian, Sun, Li, and Zhou. In each of the following three cases, determine whether it is possible to select three students to serve as group leaders, one for each group:

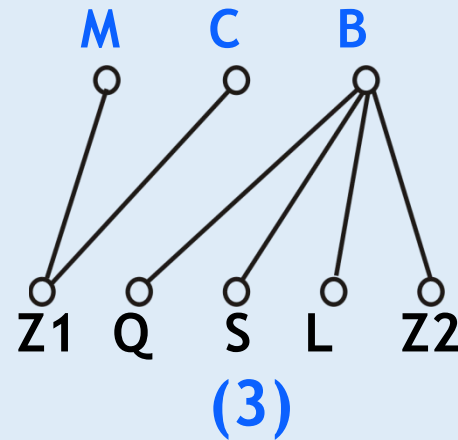
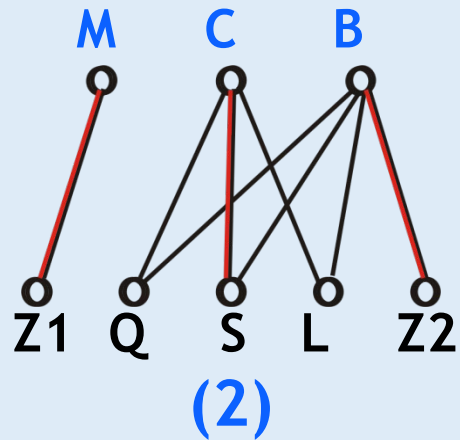
- (1) Zhao and Qian are members of the Math Group; Zhao, Sun, and Li are members of the Computer Group; Sun, Li, and Zhou are members of the Biology Group.
- (2) Zhao is a member of the Math Group; Qian, Sun, and Li are members of the Computer Group; Qian, Sun, Li, and Zhou are members of the Biology Group.
- (3) Zhao is a member of both the Math and Computer Groups; Qian, Sun, Li, and Zhou are members of the Biology Group.

M: Math Group
C: Computer Group
B: Biology Group
Z1: Zhao
Q: Qian
S: Sun
L: Li
Z2: Zhou



A complete matching corresponds to a feasible assignment.

(1) and (2) admit complete matchings, with multiple possible assignments.



(3) does not satisfy the distinctness condition, so no complete matching exists.

■ 6.4.1 Bipartite Graphs

Necessary and sufficient conditions for a graph to be bipartite
Matching, maximal matching, maximum matching, complete matching, perfect matching

■ 6.4.2 Eulerian Graphs

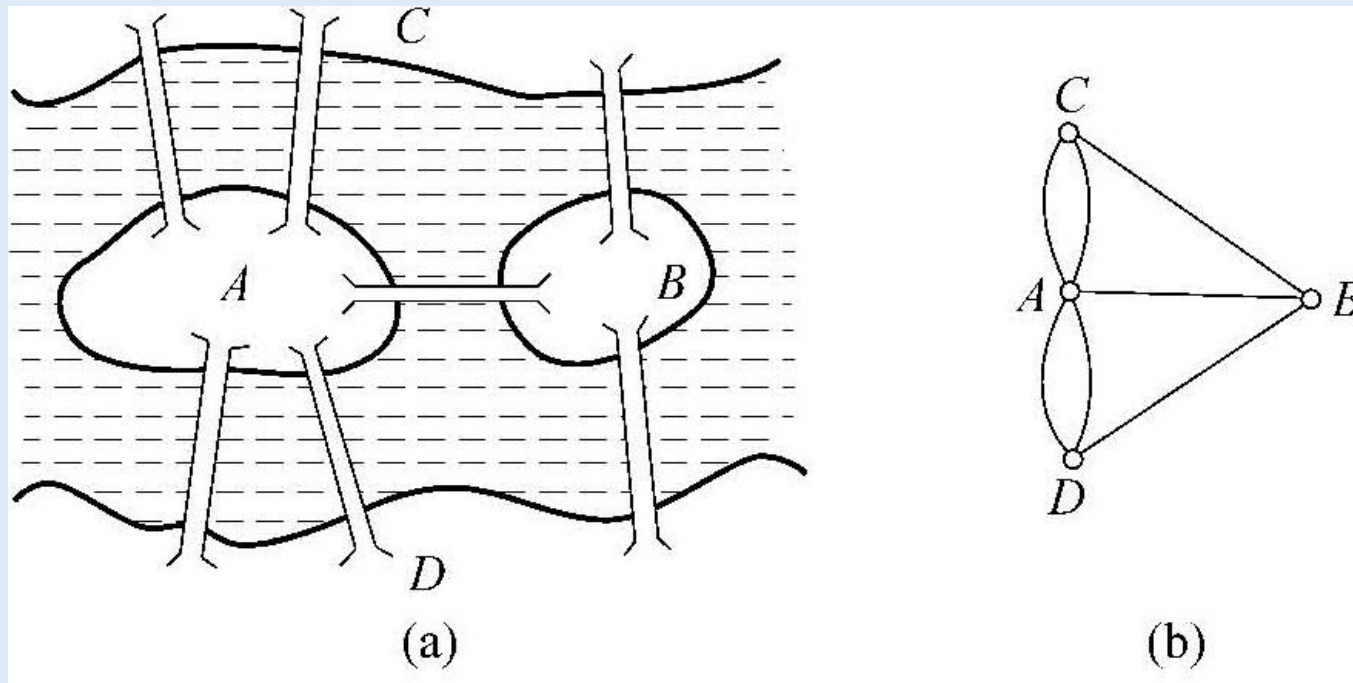
Eulerian circuits (paths) and their necessary and sufficient conditions for existence

■ 6.4.3 Hamiltonian Graphs

Hamiltonian circuits (paths) and the necessary and sufficient conditions for their existence

■ 6.4.4 Planar Graphs

Diagram of the Seven Bridges of Königsberg



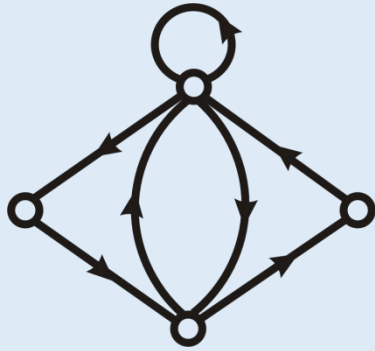
- **Eulerian Path**: A path that passes through all vertices and **each edge exactly once**.
- **Eulerian Circuit**: A circuit that passes through all vertices and **each edge exactly once**.
- **Eulerian Graph**: A graph that contains an Eulerian circuit.
- **Notes**:
 - The above definitions apply to both **undirected** and **directed** graphs.
 - A **trivial graph** is considered an Eulerian graph.
 - An Eulerian path is a **simple path**, and an Eulerian circuit is a **simple circuit**.
 - **Loops (self-edges)** do not affect the Eulerian property of a graph.

■ Theorem 6.8:

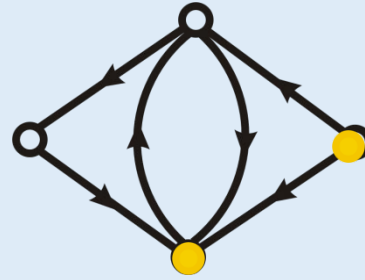
- (1) An undirected graph G has an *Eulerian circuit* if and only if G is connected and has no vertices of odd degree.
- (2) An undirected graph G has an *Eulerian path* but not an Eulerian circuit if and only if G is connected and has exactly two vertices of odd degree, with all other vertices having even degree. These two odd-degree vertices are the endpoints of every Eulerian path.

■ Theorem 6.9:

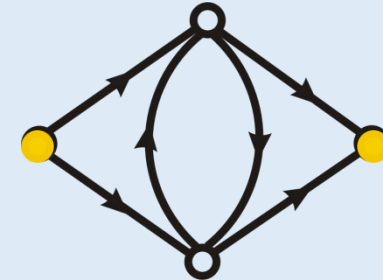
- (1) A directed graph D has an *Eulerian circuit* if and only if D is connected and the **in-degree equals the out-degree** for every vertex.
- (2) A directed graph D has an *Eulerian path* but not an Eulerian circuit if and only if D is connected and there is **one vertex** whose in-degree exceeds its out-degree by 1, and **one vertex** whose out-degree exceeds its in-degree by 1, with all other vertices having equal in-degree and out-degree.



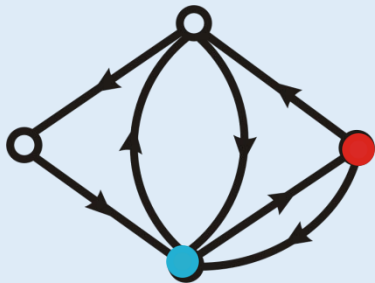
Eulerian Graph



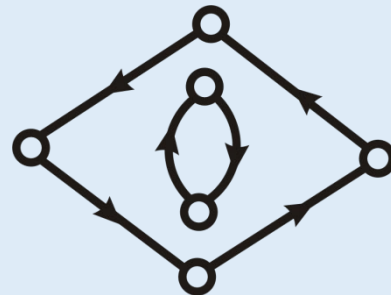
No Eulerian Path



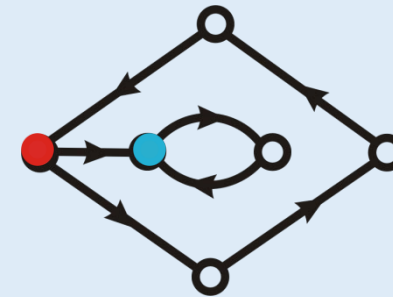
No Eulerian Path



Eulerian Path,
not Circuit



No Eulerian
Path



Eulerian Path,
not Circuit

■ 6.4.1 Bipartite Graphs

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■ 6.4.2 Eulerian Graphs

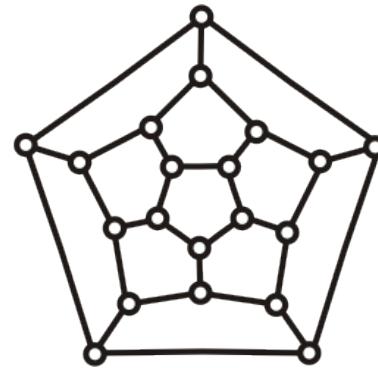
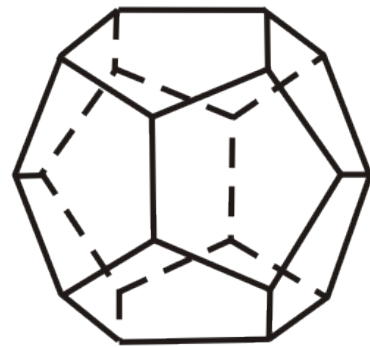
Eulerian circuits (paths) and their necessary and sufficient conditions for existence

■ 6.4.3 Hamiltonian Graphs

Hamiltonian circuits (paths) and the necessary and sufficient conditions for their existence

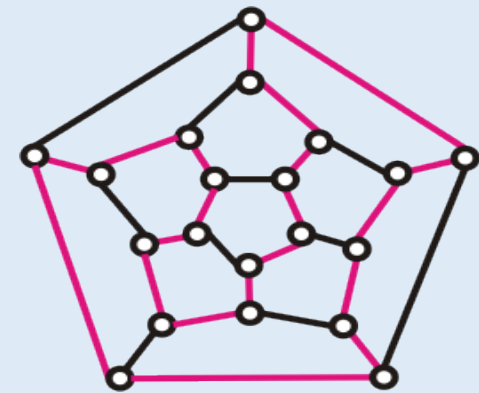
■ 6.4.4 Planar Graphs

W.Hamilton, 1859



↳ Hamiltonian Path (Circuit) and Hamiltonian Graph

- **Hamiltonian Path**: A path that visits every vertex in the graph exactly once.
- **Hamiltonian Circuit**: A circuit that visits every vertex in the graph exactly once.
- **Hamiltonian Graph**: A graph that contains a Hamiltonian circuit.
- **Notes:**
 - A Hamiltonian path is a **elementary** path.
 - A Hamiltonian circuit is a **elementary** circuit.
 - A graph having a Hamiltonian path does **not necessarily** have a Hamiltonian circuit.
 - **Loops and parallel edges do not affect** the Hamiltonian property of a graph.



■ **Theorem 6.10:** If an undirected graph $G = \langle V, E \rangle$ is a Hamiltonian graph, then for any non-empty proper subset $V_1 \subset V$, the number of connected components in $G - V_1$ satisfies:
 $p(G - V_1) \leq |V_1|$.

■ **Proof:**

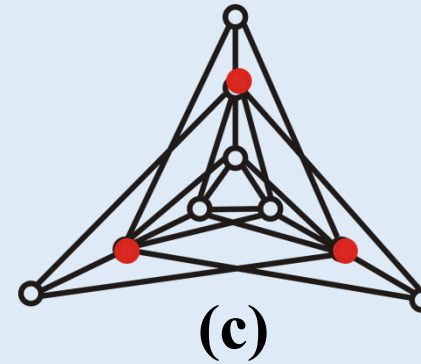
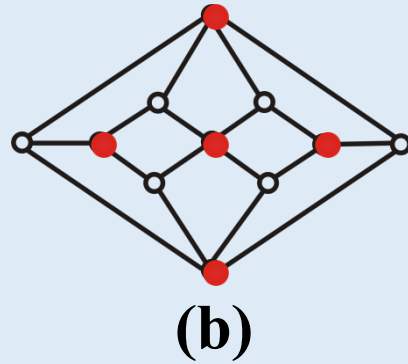
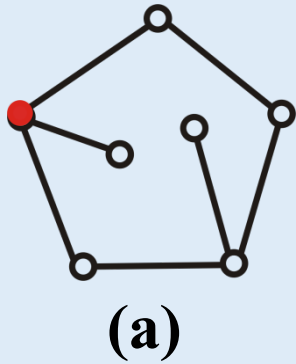
Let C be a Hamiltonian circuit in G . Then: $p(C - V_1) \leq |V_1|$,
since $p(C - V_1) \leq |V_1|$. And because $C \subseteq G$, Hence, $p(G - V_1) \leq p(C - V_1) \leq |V_1|$.

■ **Corollary:**

A graph with a **cut vertex** is not a Hamiltonian graph.

↳ Necessary Condition for Hamiltonian Graphs (e.g.)

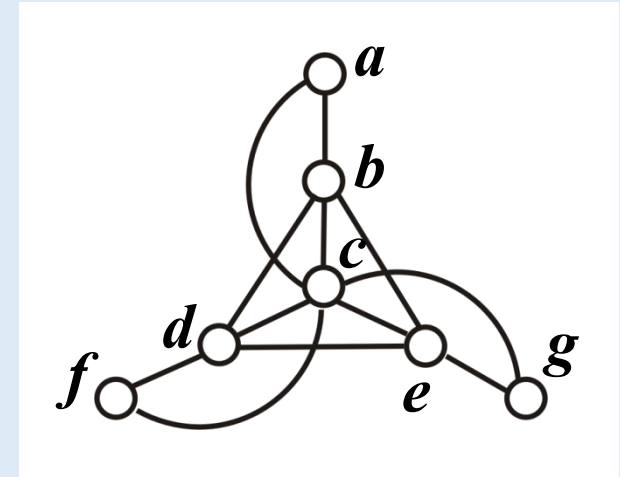
- **Example:** Prove that each of the following graphs is not a Hamiltonian graph.



There exists a Hamiltonian path in (c).

↳ Necessary Condition for Hamiltonian Graphs (e.g.)

- **Example:** Prove that the graph on the right is not a Hamiltonian graph.
- **Proof:**
 - Assume there exists a Hamiltonian circuit. a, f, g are the node of degree 2, edge (a, c) , (f, c) and (g, c) must all be included in the Hamiltonian circuit. As a result, vertex c would appear three times, which is a **contradiction**.
 - Moreover, the graph satisfies the condition of **Theorem 6.10**, which shows that the condition is necessary but not sufficient. The graph **has a Hamiltonian path**.



↳ Sufficient Conditions for Hamiltonian Graphs (Undirected Case)

■ Theorem 6.11:

Let G be a simple undirected graph of order n ($n \geq 3$).

- If the sum of the degrees of any two non-adjacent vertices is at least $n-1$, then G contains a *Hamiltonian path*.
- If the sum is at least n , then G contains a *Hamiltonian circuit*, i.e., G is a *Hamiltonian graph*.

■ Corollary:

- Let G be a *simple undirected graph* of order n ($n \geq 3$), If $\delta(G) \geq n/2$, then G is a Hamiltonian graph.
- When $n \geq 3$, the *complete graph* K_n is Hamiltonian; when $r=s \geq 2$, the *complete bipartite graph* $K_{r,s}$ is Hamiltonian.

↳ Detecting Hamiltonian Paths via Complete Underlying Graphs

■ Theorem 6.12:

Let D be a **directed graph** of order n ($n \geq 2$).

If the underlying undirected graph (obtained by ignoring the directions of all edges) contains a subgraph K_n , then D contains a **Hamiltonian path**.

■ Example: There are 7 people:

- A speaks English.
- B speaks English and Chinese.
- C speaks English, Italian, and Russian.
- D speaks Japanese and Chinese.
- E speaks German and Italian.
- F speaks French, Japanese, and Russian.
- G speaks French and German

Can they be seated around a round table so that **each person can communicate with both neighbors?**

■ Solve:

- (1) Construct an undirected graph where each person is a vertex, and there is an edge between two people **if and only** if they speak a common language.
- (2) **ACEGFDBA** is a *Hamiltonian circuit*; they can be seated in this order around the table.

